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Math-by-Mail Correspondence Program

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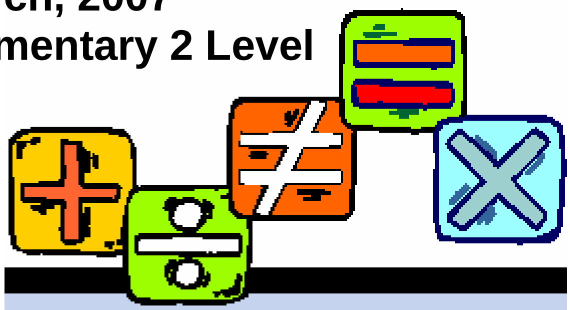
Elementary 2 Level



Logic

- Please write your answers in the answer pages at the end of the questionnaire.
- Please return the answer pages within three weeks.
- Don't worry if you did not answer all the questions – some of the questions are quite difficult. In any case, return the answer pages even if you answered only some of the questions.
- The symbol (*) preceding a question means a difficult question.
- Don't forget to fill in your email address on the answer page.

Logon to the internet club that accompanies the program, and correspond with us and the other participants via the Math By Mail forum!
http://www.weizmann.ac.il/zemed/english/hugim

If you have any questions, you can always contact us at:
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Logic

"I know what you're thinking about," said Tweedledum, "but it isn't so."

"On the contrary," continued Tweedledee, "if it was so, it might be; and if it were so, it would be: but as it isn't, it ain't. That's logic."[1]

"Let's play the true or false game," said Trudy to her twin sister, Faith, "it's very simple, anything I say is true, and anything you say is false!"

"What kind of game is that?" asked Faith, "what's the point of it?"

"The thing is that if I lie and you don't notice it, then you lose, and if you tell the truth and I don't notice, I lose," replied Trudy.

"That still seems like a foolish game to me," remarked Faith, "but let's play. I'll start – The sun shines at night."

"My turn," said Trudy, "one times one equals one."

"Okay, one times one equals zero. Maybe we can stop playing this stupid game?" Faith was bored.

"Listen," said Trudy, "if you don't stop lying and you'll always tell the truth, then I will always tell the truth! Or, you know what, if you don't say that you can't not say the truth, then I'll always lie!"

Faith was stunned. She had no clue whether Trudy was telling the truth or not. What do you think? Hopefully you will be able to answer this by the end of the questionnaire…

In our day-to-day lives, we often stumble across "true or false" problems. You may not realize that these problems are part of a very large, exciting field in Mathematics called "Logic”.

The connection between logic and math may not be immediately apparent, but once you think about it a bit (during this questionnaire you will be doing a great deal of thinking…) you might realize that many aspects of math are based on logical arguments. We begin this questionnaire with the basics of mathematical logic: true or false statements.

[1] "Through the Looking Glass" by Lewis Caroll, the author of "Alice's Adventures in Wonderland"

There are sentences in English that may be **true** or **false**.

An example of a **true** sentence: The sun shines in the daytime.
An example of a **false** sentence: The sun shines at night.
A sentence which you cannot determine to be true or false without additional information is unknown. For example: My friend went to the movies.

We use the term "truth value" to say whether a sentence is true or false.
The truth value of a sentence is **false** when the sentence is false and **true** when the sentence is true.

One of the basic concepts in logic is: **negation**. It is hard to define this in a clear and easy way, so we will try doing this using a few examples:

- "It does not rain in July" is the **negation** of the sentence "It rains in July".
- "One plus one does not equal two" is the **negation** of the sentence "One plus one equals two."

The truth value of the negation of a sentence is **false** when the sentence is **true**. The truth value of the negation of a sentence is **true** when the sentence is **false**.
Example: The truth value of the negation of "One plus one equals two" is **false** (because the sentence is true).

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1. What is the negation of each of the sentences in the answer pages? For each sentence and its negation, write if it is true, false or unknown.

An interesting thing about negation is that, in some cases, when you use it twice, repetitively, the sentence suddenly has a *positive* meaning! Even a three year old understands that his parents **want** him to finish his food when they say to him "I do not want you not to finish your food!"

And, as Sherlock Holmes said to Dr. Weston: "When you have eliminated the impossible, whatever remains, *however improbable*, must be the truth."

2. Observe the following sentence that has three negations:
"I'm not saying that there aren't programs that are unsuitable for children before nine o'clock."
a. Rewrite the sentence above so that it will be a bit more positive.
b. In general, how many negations should there be in a sentence to ensure that the meaning of the sentence is positive?

We end this section with a subtle joke…
A professor is lecturing to his class: "In English, a double negative statement gives an expression a positive meaning ("not not" means "yes"). In some languages, like Russian, a double negative statement stays negative. But, in general there is no language where a double positive statement gives an expression a negative meaning".
A voice was heard from the corner of the room: "Yeah, sure…"

"And" and "Or"

Up to now we have discussed simple sentences. Let's explore some compound sentences (a compound sentence is a sentence composed of at least two independent clauses) and see if they are true or false.

Suppose we have two sentences, sentence **a** and sentence **b** (the meanings of the sentences don't have to be logically connected).

- We say that the sentence "**a and b**" is a true sentence if both **a** and **b** are true sentences.

  Example: sentence **a**: "There are clouds in the sky when it rains." (true)

  sentence **b**: "Puddles are created when it rains." (true)

  The sentence: "There are clouds in the sky when it rains **and** puddles are created when it rains" is true.

- We say that the sentence "**a or b**" is a true sentence if at least one of the sentences **a** or **b** is a true sentence.

  Example: sentence **a**: "There are clouds in the sky when it rains." (true)

  sentence **b**: "A cat has three legs." (false)

  The sentence: "There are clouds in the sky when it rains **or** a cat has three legs" is true.

NOTE: This is the place to clarify that there is a difference between ***formal*** logic and whether or not a sentence makes sense in a spoken language. ***Formal*** logic is a set of mathematical rules that can be applied to any sentence or statement, regardless of the meaning of the sentence. A conclusion can then be made as to whether the sentence is true or false.

There are two different things that we can say about the following sentence: "There are clouds in the sky when it rains **or** a cat has three legs". First, the sentence does not make much sense. Secondly, mathematically this sentence is true (according to the rules).

Remember that we are learning math – which means we are determining when a sentence is true or false regardless of the *meaning* of the sentence.

3. For each of the sentences in the answer pages, write if it is true or false. (Note that there are no "unknown" sentences.)

Conditional Clauses

Another type of compound sentence is a conditional clause: If… then…

Once again, suppose we have two sentences: sentence **a** and sentence **b** (the <u>meanings</u> of the sentences don't have to be logically connected).

Consider the compound sentence "If sentence **a**, then sentence **b**".

Here is an example of this type of sentence: "**If** snow is white, **then** grass is green."

Now we will check the truth value of the sentence. In order to do this, we need to distinguish between a few possibilities:

- When sentence **a** is a true sentence, and sentence **b** is also a true sentence, then the conditional clause 'if **a** then **b**' is a true sentence.

  Example: sentence **a** – Snow is white. (a true sentence)

  sentence **b** – Grass is green. (a true sentence)

  The conditional clause 'if **a** then **b**' – If snow is white then grass is green. (a **true** sentence <u>according to the rule above</u>)

Important note: As before, remember that we are discussing mathematical rules, and not the logical meanings of sentences in English. Therefore, from a mathematical point of view, <u>there is no need for a logical connection between the two parts of the conditional clause, even though it may sound strange</u>!

For example, the sentence – "If one plus one equals two (true), then there are clouds in the sky when it rains (true)" is a true sentence, even though there is no logical connection between its two parts.

- When sentence **a** is a true sentence, and sentence **b** is a false sentence, then the conditional clause 'if **a** then **b**' is a false sentence.

  Example: sentence **a** – One plus one equals two. (a true sentence)

  sentence **b** – One plus two equals four. (a false sentence)

  the conditional clause 'if a then b' – If one plus one equals two then one plus two equals four. (a **false** sentence <u>according to the rule above</u>)

- When sentence **a** is a false sentence, then the conditional clause 'if **a** then **b**' will always be a true sentence, regardless of the truth value of sentence **b**. (Even though this sounds very strange!)

  Example: sentence **a** – One plus one equals three. (a false sentence)
  sentence **b** – Two plus two equals four. (a true sentence)
  The conditional clause 'if **a** then **b**' – If one plus one equals three then two plus two equals four. (a **true** sentence because the first sentence was **false**, even though the second sentence is true!)

4. For each of the sentences in the answer pages, write if it is true or false (there are no "unknown" sentences).

If and Only If (= iff)

There is another type of conditional clause, which is constructed as follows: sentence **a** *if and only if* sentence **b** (as before, a and b donate sentences).

Here is an example of this type of sentence: "Aladdin married Jasmine *if and only if* Jasmine married Aladdin."

Once again, we will check the truth value of the sentence. In order to do this, we need to distinguish between a few possibilities:

- The sentence '**a** if and only if **b**' is a true sentence in the following two cases:

  a. Both sentence **a** and sentence **b** are true sentences.

  Example: sentence **a** – Snow is white. (a true sentence)
  sentence **b** – Grass is green. (a true sentence)
  The conditional clause '**a** if and only if **b'** – Snow is white if and only if grass is green. (a **true** sentence <u>according to the rule above</u>)

  b. Both sentence a and sentence b are false sentences.

  Example: sentence **a** – Bananas are always purple. (a false sentence)
  sentence **b** – Tomatoes grow on tall trees. (a false sentence)
  The conditional clause '**a** if and only if **b'** – Bananas are always purple if and only if tomatoes grow on tall trees. (a **true** sentence <u>according to the rule above</u>)

- The sentence '**a** if and only if **b**' is a false sentence when one of the two sentences is a true sentence, and the other is a false sentence (the order of the sentences doesn't matter[2]).

  Example: sentence **a** – Two is greater than three. (a false sentence)
  sentence **b** – Four is an even number. (a true sentence)
  The conditional clause '**a** if and only if **b'** – Two is greater than three if and only if four is an even number. (a **false** sentence <u>according to the rule above</u>)

[2] It doesn't matter if the first one is true and the second one is false, or the opposite.

5. a. Write two examples of true 'if and only if' conditional clauses (one sentence for each instance), and an example of a false 'if and only if' conditional clause.

b. Explain (you can do this by using an example) what is the difference between the two types of conditional clauses ('if sentence a then sentence b' vs. 'sentence a if and only if sentence b').

Truth Tables

We often use tables called "truth tables" to write the truth values of sentences.

Truth tables are constructed as follows:

- Each column in the table is called a column of truth values.
- In each cell we write "true" or "false" according to the rules we have just learnt.

For example, here is a truth table for the "and" connective:

Sentence a	Sentence b	Sentence 'a and b'
true	true	true
true	false	false
false	true	false
false	false	false

6. Fill in the truth table in the answer pages.

Observe the truth table you have just filled in.

7. Do you see a (compound) sentence in the table whose truth values are all "true" (in other words, all the values in this sentence's column are "true")? If your answer is yes, write down this sentence.

A sentence whose truth values in the truth table are all "true" is called a **tautology**.

---

8. Do you see a (compound) sentence in the table whose truth values are all "false" (in other words, all the values in this sentence's column are "false")? If your answer is yes, write down this sentence.

A sentence whose truth values in the truth table are all "false" is called a **contradiction**.

Deduction

Many logical problems can be solved by making conclusions from a set of statements or facts. These questions can sometimes be solved using tables or graphs.

Here is an example: Tom, Jerry and Garfield played in a band. Each one of them played a different musical instrument (a guitar, a violin and a flute). Using the following statements, find which instrument each one played:

a. Tom did not play a wind instrument.
b. One of the musicians played an instrument that starts with the same letter as his name.

<u>Solution</u>:

First we will write all the possibilities for each player in a table:

Tom	**Jerry**	**Garfield**
guitar	guitar	guitar
violin	violin	violin
flute	flute	flute

As a result of the first statement, we will erase the flute from Tom's possible musical instruments:

Tom	**Jerry**	**Garfield**
guitar	guitar	guitar
violin	violin	violin
~~flute~~	flute	flute

The second statement leads us to the conclusion that Garfield played the guitar (both Garfield and guitar start with a 'g'), so we can erase the violin and the flute from Garfield's list. We can also erase the guitar from Tom's list (since Garfield is playing it) – which implies that he plays the violin, and the problem is solved since we are left with only one possibility: Jerry plays the flute.

Tom	**Jerry**	**Garfield**
~~guitar~~	~~guitar~~	<u>**guitar**</u>
<u>**violin**</u>	~~violin~~	~~violin~~
~~flute~~	<u>**flute**</u>	~~flute~~

9. Solve the following riddles (using the technique above – or any other method you choose):

a. Three students - Abraham, Isaac and Jacob wrote a science project. When they handed it in, the teacher found a serious calculation error in the second part of the project. The teacher wanted to find out who made the mistake and called in the students one by one. Each of the students made a statement as follows:

Abraham – I did not make a mistake, Isaac did, and in any case, I wrote a different part of the project.

Isaac – Jacob made a mistake and I know how to correct it. Even great scientists make mistakes sometimes.

Jacob – I did not make a mistake, and I've been thinking that something is wrong for a while. By the way, Abraham did write a different part of the project.

The teacher knew that one of the sentences that each student said is true, and the other is false. Which student made the mistake in the second part of the project?

b. Joan, Lorie and Sherry each have two professions. The professions are: an author, an architect, a teacher, a doctor, a lawyer and an artist. Find the two professions of each woman using the following statements (every character in each sentence is a different woman):

- The teacher and the author went to visit Joan.
- The doctor ordered a painting from the artist.
- The doctor met the teacher.
- The artist is a relative of the architect.
- Sherry won chess games against Lorie and the artist.
- Lorie is a neighbor of the author.

Is there more than one solution to this problem?

c. David, Solomon, Esther and Miriam live on a farm, and each has a dog and a cat. Every person named his dog after one of his three friends, and his cat after one of his two other friends. There are no two cats, nor two dogs with the same name.

Miriam's dog and Esther's cat are named after the owner of the cat called "Esther".

Solomon's cat' name is identical to the name of the person whose cat is named after the owner of the dog "David". Who is the owner of the dog "David"?

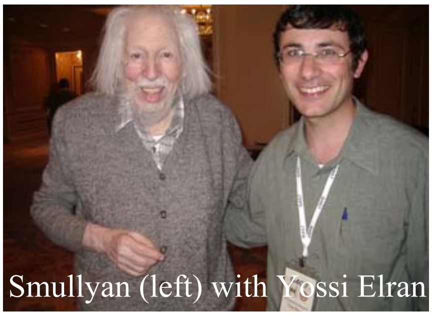
The Logic of Raymond Smullyan

Raymond Smullyan is one of the most creative people in the field of logic puzzles. He was born in 1919 in New York, and started out as a professional magician, performing all over the United States. He went on to study math at Princeton University.

He is known as one of the important logicians alive today. Although he has made important contributions to Godellian Logic (named after another famous mathematician – Kurt Godel), he has remained a magician at heart!

Smullyan has written excellent books on logic puzzles, through which he managed to teach some of his important math discoveries (in logic) in a way almost anyone could understand. Some of his famous books are: "What is the Name of this Book?" and "The Lady or the Tiger?"

Smullyan is still active, and we met him recently in a conference on math and magic that was held in Atlanta, U.S.A. He is a very friendly man who is constantly smiling and has a riddle ready for whoever he meets. Maybe that is the reason that the chairman introduced him as: "Raymond Smullyan who will prove that either he doesn't exist or you don't exist, but you will never know which!"


Smullyan (left) with Yossi Elran

Smullyan has greatly enriched the field of logic puzzles. His favorite puzzles are tales of "knights" (people that always tell the truth) and "knaves" (people that always lie). One such puzzle involves two cities: one city of "knights", and the other of "knaves". The residents of the two cities like to visit each other. One day, a man enters one of the cities, but doesn't know which. He decides to ask the first person he meets where he is. The answer he receives is that he is in the "knaves" city. A few seconds later, the visitor realizes that he still doesn't know where he is. He *may* be in the "knaves" city, and the person he met was a "knight" visiting there, but he also may be in the "knights" city and the person he met was a "knave" visiting there! Therefore, the visitor decides to ask another question. What question can he ask in order to know definitely which city he is in?

- The answer to this problem is that he should ask: "Do you live here?"

Explanation: If the visitor is in the "knights" city, then a "knight" will answer YES, and a "knave" (visiting there) will also answer YES (remember that "knaves" always lie!). On the other hand, if the visitor is in the "knaves" city, both a "knight" and a "knave" will answer NO. So, if the visitor asks a person "do you live here?" and receives a positive answer, he will know that he is in the "knights" city, and if he receives a negative answer, he will know that he is in the "knaves" city.

In "The Lady or the Tiger", Smullyan describes an imaginary island called "The Isle of Dreams", whose residents are always awake during the day, and always sleep at night. When they sleep, they dream dreams that seem very realistic, and are a continuation of whatever happened during the day. When they wake up, their real lives seem to be a continuation of their dream life, so that sometimes they don't know if they are asleep or awake!

The residents of the island are of two types:

- People Smullyan calls: Diurnals – that believe that whatever happens when they are awake is true, and whatever happens when they are asleep is false.
- People Smullyan calls: Nocturnals – that believe that whatever happens when they are awake is false, and whatever happens when they are asleep is true.

Now it is your turn to try and answer a few of Smullyan's logic puzzles!

10. Explain why the sentences in the answer pages are true.

11. Once one of the residents thought that he was nocturnal.

a. Was he right?

b. Was he awake or asleep at that time?

12. One of the residents of "The Isle of Dreams" thought that he was both diurnal and asleep. What was he really? (Important note: the sentence is a compound "and" sentence)

Raymond Smullyan calls the following riddle a "Metapuzzle":

13. a. One day I met a friend and asked him the following riddle:
A resident of "The Isle of Dreams" thought that he was both diurnal and awake. What was he really?
My friend remarked that he can't answer this question, and asked me – do you know the answer to your question? (If this resident was diurnal/nocturnal, awake/asleep.)
I answered that I did know (and I always tell the truth!!).
My friend continued and asked me: If you tell me if this resident was diurnal or nocturnal, would I be able to know if he was asleep or awake? I answered (truly) that yes, he would be able to know the true answer.
If that is the case, my friend said, I can solve your riddle. What was the answer that my friend found?
(Important note: the sentence is a compound "and" sentence)
b. Explain why Raymond Smullyan calls this a "Metapuzzle".

14. Return to Trudy and Faith's logic game. Was Trudy's sentence true or false? (Hint: in order to answer this question, try breaking up the compound sentence to small simple sentences).

The next questionnaire will be: Ways to weigh…

Bibliography and Resource Material

- **Original work of Michal Elran and Dr. Yossi Elran**
- **Rob Eastaway and Jeremy Wyndham, "Why do Buses Come in Threes? The Hidden Mathematics of Everyday Life"**
- **Raymond Smullyan, "What is the Name of this Book?"**
- **Raymond Smullyan, "The Lady or the Tiger? And Other Logic Puzzles"**
- **Caroll Lewis, "Through the Looking Glass"**

From the web:

http://www.cs.odu.edu/~jzhu/courses/content/logic/intr_to_logic.html

http://en.wikipedia.org/wiki/Raymond_Smullyan

http://learn.snunit.k12.il/snunit/lashon/upload/teachers/Alicesyn.doc (a Hebrew webiste)

<u>Answer page – elementary 2</u>

First Name____________________ Last name_____________ male/female

Birth date_______________________ Tel. Number ____________________

Address ____

School: _______________________________ Grade __________________

Email ____

Remember you can submit your answers on line!

Simply enter your username and password on our website:

http://www.weizmann.ac.il/zemed/english/hugim

and the answers will be e-mailed to you shortly!

1. For each sentence and its negation sentence, write if it is true/false/unknown.

a. It is cold in the winter in Canada. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

b. Iraq is a neighbor of Israel. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

c. **Every** even number ends with a four. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

d. **Every** even number that ends with a four is even. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

e. He never doesn't tell the truth. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

f. There are eight days in a week. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

g. An octagon has only seven sides. true / false / unknown

The negation sentence is:

_______________________________________ true / false / unknown

2. a. The sentence "I'm <u>not</u> saying that there are<u>n't</u> programs that are <u>un</u>suitable for children before nine o'clock" rewritten in a more positive manner is:

____

____

b. In general, the number of negations there should be in a sentence so that the meaning of the sentence will be positive is: ________________________________

3. For each of the following complex sentences, write if it is true or false.

a. 1 plus 1 equals 2 **<u>and</u>** 2 plus 3 equals 4.	true / false
b. An apple can be red **<u>or</u>** a banana is purple.	true / false
c. A spider has six legs **<u>and</u>** a cow has four legs.	true / false
d. Kids like candy **<u>or</u>** there are clouds when it rains.	true / false
e. Thirty is divisible by 6 **<u>or</u>** thirty is divisible by 10.	true / false
f. Thirty is divisible by 7 **<u>and</u>** thirty is divisible by 10.	true / false
g. Zero is divisible by 2 **<u>or</u>** two is divisible by zero.	true / false
h. I am solving a questionnaire now **<u>and</u>** I lied in the first part of this sentence.	true / false

4. For each of the following conditional clauses, write if it is true or false.

a. **<u>If</u>** it is cold in the winter in Canada **<u>then</u>** it is hot in the summer in Canada.	true / false
b. **<u>If</u>** there are green apples **<u>then</u>** there are grey bananas.	true / false
c. **<u>If</u>** a dog has six legs **<u>then</u>** a cow has four legs.	true / false
d. **<u>If</u>** two is an even number **<u>then</u>** a cow has four legs.	true / false
e. **<u>If</u>** two is an odd number **<u>then</u>** the number 731 is divisible by 2.	true / false
f. **<u>If</u>** the moon is made out of cheese **<u>then</u>** there are craters in the moon.	true / false

5. a. Examples of true 'if and only if' conditional clauses (one sentence for each instance):

1)____

2)____

An example of a false 'if and only if' conditional clause:

____

b. An explanation of what is the difference between the two types of conditional clauses:

____

____

____

6. Fill in the following truth tables (when we are talking about the negation of sentence a we will write not-a, and iff = if and only if):

A	b	'a or b'	Not-a	'if a then b'	'a iff b'	'not-a or a'	'not-a and a'	'not-a or b'
True	true							
True	false							
False	true							
False	false							

7. (choose the correct answer) I do / don't see a (compound) sentence in the table whose truth values are all "true".

If the answer was positive, write the title of this column: ______________________________

8. (choose the correct answer) I do / don't see a (compound) sentence in the table whose truth values are all "false".

If the answer was positive, write the title of this column: ______________________________

9. a. The student that made the mistake in the second part of the science project was:

____

b. Joan is a ____________________________ and a ____________________________.

Lorie is a ____________________________ and a ____________________________.

Sherry is a ____________________________ and a ____________________________.

(choose the correct answer) There is / isn't more than one solution to this problem.

Explanation: ____

____

____

c. The owner of the dog "David" is: ______________________________________

10. a. Any resident that is awake believes that he is diurnal.

Explanation why this sentence is true: ____

____

b. Any resident that is asleep believes that he is a nocturnal.

Explanation why this sentence is true: ____

____

____

c. Nocturnals always believe that they are asleep.

Explanation why this sentence is true: ____

____

d. Diurnals always believe that they are awake.

Explanation why this sentence is true: ____

____

11. Once one of the residents thought that he was nocturnal.

a. Was he right? YES / NO / CAN'T TELL

Explanation: ____

____

b. Was he awake or asleep at that time? AWAKE / ASLEEP / CAN'T TELL

Explanation: ____

____

12. One of the residents of "The Isle of Dreams" thought that he was both diurnal and asleep.

This resident was:____

Explanation: ____

____

13. a. The resident in the riddle was:____

Explanation: ____

____

b. Raymond Smullyan calls this a "Metapuzzle" because: ____

____

____

14. Return to Trudy and Faith's logic game.

Trudy's sentence was: "If you don't stop lying and you'll always tell the truth, then I will always tell the truth! Or, you know what, if you don't say that you can't not say the truth, then I'll always lie!"

Trudy's sentence was true / false.

(Remember that Trudy always told the truth, and Faith always lied.)

Explanation: ____

____

Feedback

The questionnaire was:	□ very easy	□ easy	□ difficult	□ very difficult
The questionnaire was:	□ very boring	□ boring	□ interesting	□ very interesting

I answered the questions:

□ alone □ with very little help □ with some help □ with lots of help